MISG 2018: Instability in Fluids

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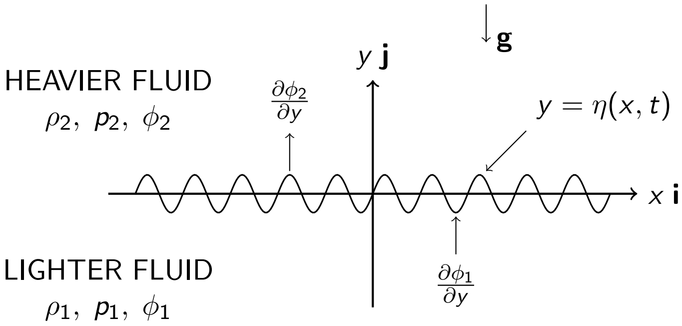
Rayleigh-Taylor Instability



Rayleigh-Taylor Instability

- The problem proposed is a situation with 2 fluids, one atop the other with different densities. Between them is the interface $\eta(x, t)$ which is a perturbation across $y = 0$.
- Some assumptions are made:
  The vorticity is 0 (it is irrotational) so $\nabla \times v = 0$
  It is incompressible, meaning the volume is constant. This results in $\nabla \cdot v = 0$

Equations of State and Boundary Conditions

- $$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0$$
  $$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0$$
  $$y = 0 : \frac{\partial \phi_1}{\partial y}(x, 0, t) = \frac{\partial \eta}{\partial t}(x, t)$$
  $$y = 0 : \frac{\partial \phi_2}{\partial y}(x, 0, t) = \frac{\partial \eta}{\partial t}(x, t)$$
  $$y = 0 : \rho_1[\frac{\partial \phi_1}{\partial t}(x, 0, t) + g\eta(x, t)] = \rho_2[\frac{\partial \phi_2}{\partial t}(x, 0, t) + g\eta(x, t)]$$

Form of Solution and Dispersion Relation

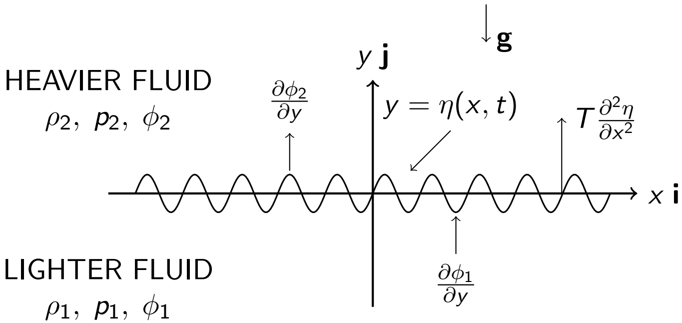
- $$\phi_1(x, y, t) = F_1(y) exp[i(kx - \omega t)]$$
  $$\phi_2(x, y, t) = F_2(y) exp[i(kx - \omega t)]$$

- $$\omega = \pm i \sqrt{\frac{kg(\rho_2 - \rho_1)}{\rho_1 + \rho_2}}$$

Solution and Analysis

- $$Re(\eta) = A_1 \cos(kx)\sqrt{\frac{k(\rho_1 + \rho_2)}{g(\rho_2 - \rho_1)}}(e^{\beta t} - e^{-\beta t})$$
- This is an unstable exponentially growing standing wave if $\rho_2 > \rho_1$, and is a stable standing wave if $\rho_1 > \rho_2$

Rayleigh-Taylor Instability with Surface Tension



Instability of Fluids with Interfacial Tension

- Net upward force per unit area due to interfacial tension

$$T\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 \phi_1}{\partial x^2}+\frac{\partial^2 \phi_1}{\partial y^2}=0$$

$$\frac{\partial^2 \phi_2}{\partial x^2}+\frac{\partial^2 \phi_2}{\partial y^2}=0$$

$$y=0:\frac{\partial \phi_1}{\partial y}(x,0,t)=\frac{\partial \eta}{\partial t}(x,t)$$

$$y=0:\frac{\partial \phi_2}{\partial y}(x,0,t)=\frac{\partial \eta}{\partial t}(x,t)$$

$$y=0:\qquad p_1(x,0,t)+T\frac{\partial^2 y}{\partial x^2}=p_2(x,0,t)$$

Form of Solution and Dispersion Relation

- The form of solution:

$$\eta(x,t) = \eta_0 exp[i(kx - \omega t)]$$

$$\phi_1(x,y,t) = F_1(y) exp[i(kx - \omega t)]$$

$$\phi_2(x,y,t) = F_2(y) exp[i(kx - \omega t)]$$

- Dispersion Relation:

$$\omega = \pm\sqrt{\frac{k}{(\rho_2 + \rho_1)}(-g(\rho_2 - \rho_1) + Tk^2)}$$

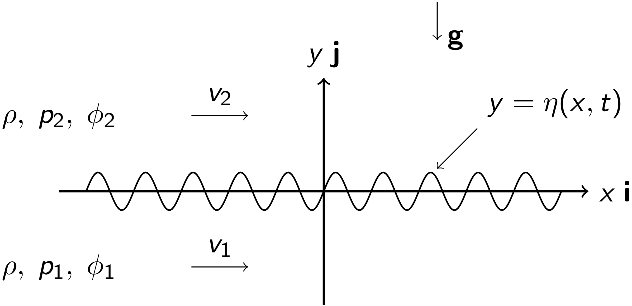
Solution and Analysis

- Stable if

$$k^2 > \frac{(\rho_2 - \rho_1)g}{T}$$

$$\lambda < \sqrt{\frac{4\pi^2 T}{(\rho_2 - \rho_1)g}} = 2\pi\sqrt{\frac{T}{(\rho_2 - \rho_1)g}}.$$

Kelvin-Helmholtz Instability



Kelvin-Helmholtz Instability

- $$V_x^{(1)} = V_1 + \frac{\partial \phi_1}{\partial x}, \qquad V_y^{(1)} = \frac{\partial \phi_1}{\partial y}$$
$$V_x^{(2)} = V_2 + \frac{\partial \phi_2}{\partial x}, \qquad V_y^{(2)} = \frac{\partial \phi_2}{\partial y}$$

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0; \qquad \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0,$$

$$\frac{\partial \phi_1}{\partial y}(x, 0, t) = \frac{\partial \eta}{\partial t} + V_1 \frac{\partial \eta}{\partial y}.$$
$$\frac{\partial \phi_2}{\partial y}(x, 0, t) = \frac{\partial \eta}{\partial t} + V_2 \frac{\partial \eta}{\partial y}.$$

$$V_1 \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial t} = V_2 \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial t}.$$

Form of Solution and Dispersion Relation

- The form of solution:

$$\eta(x,t) = \eta_0 exp[i(kx - \omega t)]$$

$$\phi_1(x,y,t) = F_1(y) exp[i(kx - \omega t)]$$

$$\phi_2(x,y,t) = F_2(y) exp[i(kx - \omega t)]$$

- Dispersion Relation:

$$\omega = \frac{k(V_2 + V_1) \pm ik(V_1 - V_2)}{2}$$

Solution and Analysis

- The perturbation solution is

$$\eta(x,t) = \eta_0 \exp\left[i\left(kx - \frac{k}{2}(V_2+V_1)t\right) - \frac{k}{2}(V_1 - V_2)t\right] +$$

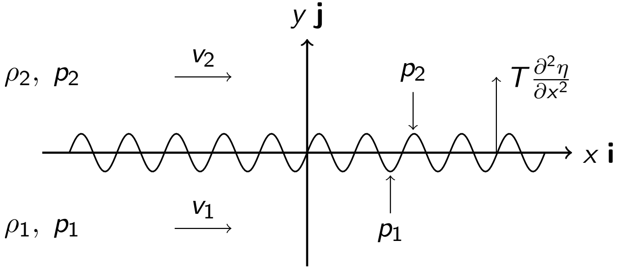
$$\eta_0 \exp\left[i\left(kx - \frac{k}{2}(V_2+V_1)t\right) + \frac{k}{2}(V_1 - V_2)t\right]$$

$$Re[\eta(x,t)] = \eta_0 \cos\left(kx - \frac{k}{2}(V_2+V_1)t\right)\left(\exp\left[-\frac{k}{2}(V_2+V1)t\right]\right.$$

$$\left. + \exp\left[-\frac{k}{2}(V_1 - V2)t\right]\right)$$

This is unstable for $V_1 < V_2$ and $V_2 < V_1$.

Kelvin-Helmholtz and Rayleigh-Taylor Instability with Interfacial Tension



Kelvin-Helmholtz and Rayleigh-Taylor Instability with Interfacial Tension

- $$V_x^{(1)} = V_1 + \frac{\partial \phi_1}{\partial x}, \qquad V_y^{(1)} = \frac{\partial \phi_1}{\partial y}$$
  $$V_x^{(2)} = V_2 + \frac{\partial \phi_2}{\partial x}, \qquad V_y^{(2)} = \frac{\partial \phi_2}{\partial y}$$
  $$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0; \qquad \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0,$$
  $$\frac{\partial \phi_1}{\partial y}(x,0,t) = \frac{\partial \eta}{\partial t} + V_1 \frac{\partial \eta}{\partial y}.$$
  $$\frac{\partial \phi_2}{\partial y}(x,0,t) = \frac{\partial \eta}{\partial t} + V_2 \frac{\partial \eta}{\partial y}.$$
  $$y = 0 : \qquad p_1(x,0,t) + T\frac{\partial^2 y}{\partial x^2} = p_2(x,0,t)$$

Form of Solution and Dispersion Relation

- The form of solution:

$$\eta(x,t) = \eta_0 exp[i(kx - \omega t)]$$
$$\phi_1(x,y,t) = F_1(y) exp[i(kx - \omega t)]$$
$$\phi_2(x,y,t) = F_2(y) exp[i(kx - \omega t)]$$

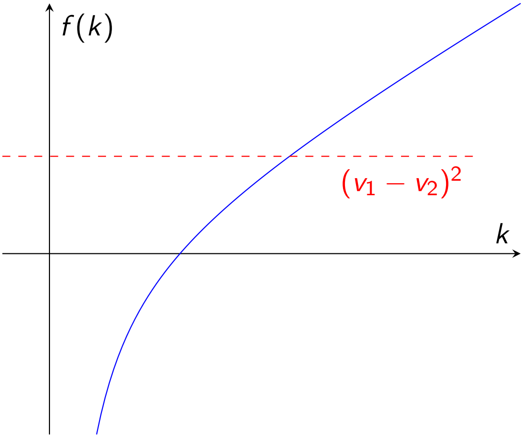
- $$\frac{\omega}{k} = \frac{\rho_1 V_1 + \rho_2 V_2 \pm \sqrt{Q}}{\rho_1 + \rho_2}$$

  where $Q = -\rho_1\rho_2(V_1 - V_2)^2 + (\rho_1 + \rho_2)(Tk - {}^{g}/_{k}(\rho_2 - \rho_1))$

Solution and Analysis

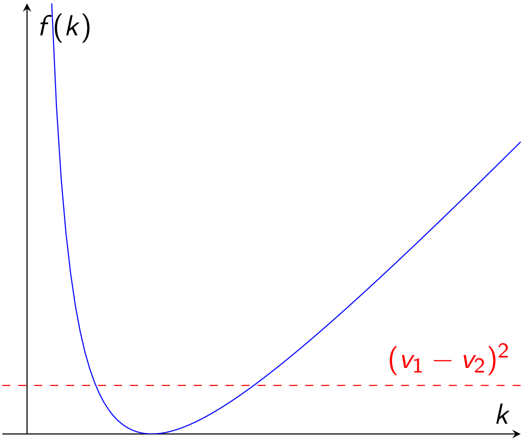
- It is unstable when $(V_1 - V_2)^2 > \frac{(\rho_1+\rho_2)(Tk - \frac{g}{k}(\rho_2 - \rho_1))}{\rho_1 \rho_2}$
- $k > \sqrt{\frac{g}{T}(\rho_2 - \rho_1)}$ is the first necessary condition for stability

Graphical Analysis for $\rho_2 > \rho_1$



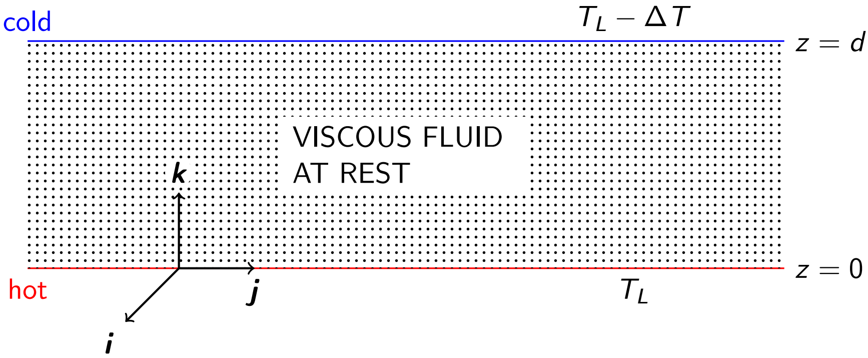
$$f(k) = \frac{(\rho_1 + \rho_2)(Tk - \frac{g}{k}(\rho_2 - \rho_1))}{\rho_1 \rho_2}$$

Graphical Analysis for $\rho_1 > \rho_2$



$$f(k) = \frac{(\rho_1 + \rho_2)(Tk + \frac{g}{k}(\rho_1 - \rho_2))}{\rho_1 \rho_2}$$

Benard Problem



Benard Problem

- We wish to study the instability in the fluid owing to heat transfer - that is, when does the heat transfer mechanism switch from conduction to convection
- Temperature Gradient: $\frac{dT}{dz} = -\frac{\Delta T}{d}$
- Navier-Stokes Momentum Eqn: $\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + \rho g$
- Energy Eqn: $\frac{DT}{Dt} = \kappa \nabla^2 T +$viscous 2nd order terms
- Equation of State: $\rho = \tilde{\rho}(1 - \alpha(T - \tilde{T}))$

Benard Problem - Unperturbed State

- $$v_0 = 0$$
  $$T = T_0(z)$$
  $$\rho = \rho_0(z)$$
  $$p = p_0(z)$$
- Simplified Constitutive Eqns:

$$\frac{dp_0}{dz} = -\rho_0 g$$

$$\kappa \frac{d^2 T_0}{dz^2} = 0$$

$$\rho_0 = \tilde{\rho}(1 - \alpha(T_0 - \tilde{T}))$$

Benard Problem - Unperturbed State

- Solving yields:

$$T_0(z) = T_L - \frac{\Delta Tz}{d}$$

$$\rho_0(z) = \tilde{\rho}(1 - \alpha(T_L - \frac{\Delta Tz}{d} - \tilde{T}))$$

$$p_0(z) = C + g\tilde{\rho}z(\alpha(T_L - \frac{\Delta Tz}{2d} - \tilde{T}) - 1)$$

Perturbation and Boussinesq Approximation

- We make the Boussinesq Approximation: we take ρ to be constant and approximately $\tilde{\rho}$ unless it gives rise to buoyancy forces in the Navier-Stokes eqn
- We perturb the state by making it no longer at rest:

$$v(x, y, z, t) = 0 + v_1(x, y, z, t)$$

$$T(x, y, z, t) = T_0(z) + T_1(x, y, z, t)$$

$$\rho(x, y, z, t) = \rho_0(z) + \rho_1(x, y, z, t)$$

$$p(x, y, z, t) = p_0(z) + p_1(x, y, z, t)$$

New Constitutive Equations

- We still have incompressibility: $\nabla \cdot v = \nabla \cdot v_1 = 0$
- $$\tilde{\rho}\frac{\partial v_1}{\partial t} = -\nabla p_1 + \mu \nabla^2 v_1 + \rho_1 g$$

$$\frac{\partial T_1}{\partial t} - v_1^{(z)} \frac{\Delta T}{d} = \kappa \nabla^2 T_1$$

$$\rho_1 = -\tilde{\rho} \alpha T_1$$

Form of Solution and Eigenvalue Problem

- We want to find $v_1^{(z)}$ to analyse the stability of the system. We assume the form

$$v_1^{(z)} = w(z)f(x,y)e^{st}$$

- We obtain an eigenvalue problem with eigenfunction w(z)

$$\left[\left(\kappa(D_z^2 - a^2) - s\right)\left(\nu(D_z^2 - a^2) - s\right)(D_z^2 - a^2) - \alpha a^2 g \frac{dT_0}{dz}\right] w(z)$$

$$= 0$$

Boundary Conditions and Form of Solution

- $w(0) = w(d) = 0$ as $v_1^{(z)}$ would be 0 at the boundary, as the boundaries are not moving.

$$\frac{d^2 w}{dz^2}(0) = \frac{d^2 w}{dz^2}(d) = 0$$
$$\frac{d^4 w}{dz^4}(0) = \frac{d^4 w}{dz^4}(d) = 0$$

- To satisfy the boundary conditions, we let

$$w(z) = \sin(\frac{n\pi z}{d}); n = 1, 2, 3..$$

Stability for Negative Temperature Gradient

- To analyse stability we need to focus on the e^{st} factor. Solving the eigenvalue problem for w(z) yields an equation for s.
- Solving $(\nu - \kappa)^2 a_*^6 + 4\alpha a^2 g \frac{\Delta T}{d} > 0$ for a yields that in this case, s is always real
- However, for stability, we need $s < 0$. This is true when

$$\frac{\alpha g d^3 \Delta T}{\nu\kappa} < \frac{27\pi^4}{4}$$

Where Rayleigh's number is defined as $R = \frac{\alpha g d^3 \Delta T}{\nu\kappa}$

Stability for Positive Temperature Gradient

- Solving $(\nu - \kappa)^2 a_*^6 - 4\alpha a^2 g \frac{\Delta T}{d} > 0$ for a yields that in this case, s is not always real.
  We require
  $$\frac{\alpha g \Delta T d^3}{(\nu - \kappa)^2} > \frac{27\pi^4}{16}$$
  in order for s to be real, otherwise we get an oscillating velocity, where Mason's number is defined as
  $M = \frac{\alpha g \Delta T d^3}{(\nu - \kappa)^2} > \frac{27\pi^4}{16}$
- If s is real, for stability, we need $s < 0$. This is true when
  $$\frac{\alpha g d^3 \Delta T}{\nu\kappa} > -\frac{27\pi^4}{4}$$
  Which is always true, as the Rayleigh number is always positive

Convective Unstable Fluid